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EXPLICIT FORMULA FOR RECIPROCAL GENERATING FUNCTION AND ITS APPLICATION

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ABSTRACT. **In this work, we find a new explicit formula for the coefficients of a reciprocal generating function $G(x)$ raised to the power of α and apply the obtained results for getting new identities, particularly, for the Norlund polynomial and the Bernoulli numbers of the second kind. Also we establish a new explicit formula for obtaining coefficients of a compositional inverse generating function.**

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1. INTRODUCTION

By a reciprocal generating function we mean the following [1]:

Definition 1.1. A reciprocal generating function $G(x)$ of a generating function $A(x) = \sum\limits_{n\geq 0} a(n)x^n$ is a formal power series such that satisfies the following condition:

$$A(x) \times G(x) = 1, \tag{1}$$

that is

$$G(x) = \frac{1}{A(x)}.$$

In the paper [2] authors found a method for obtaining the coefficients of a compositional inverse generating function and a reciprocal generating function $xA(x)$. This method is based on the notion of a composita that is defined in [3, 4]. Recently, this topic has also been studied in [5], where a method using free cumulants to compute the coefficients of a reciprocal generating function was presented.

The main aim of the paper is to find a new explicit formula for the coefficients of a reciprocal generating function $G(x)$ raised to the power of α and apply the obtained results for getting new identities and a new explicit formula for obtaining the coefficients of a compositional inverse generating function.

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2. Main results

Theorem 2.1. *Suppose for a generating function* $A(x) = \sum_{n\geq 0} a(n)x^n$ *with* $a(0) \neq 0$ *the coefficients of* $A(x)$ *raised to the power of* k *are defined by*

$$a(n,k) = [x^n]A(x)^k.$$

Then an explicit formula for the coefficients of a reciprocal generating function $G(x)$ *raised to the power of* α*, that is*

$$G(x,\alpha) = \sum_{n\geq 0} g(n,\alpha)x^n = \left(\frac{1}{A(x)}\right)^\alpha,$$

is equal to

$$g(n,\alpha) = \sum_{k=0}^{n} a(0)^{-k-\alpha}(-1)^k \binom{k+\alpha-1}{k}\binom{n+\alpha}{n-k} a(n,k). \tag{2}$$

Proof. Firstly we consider the case for $a(0) = 1$. From the definition of a reciprocal generating function (1), a generating function $G(x,\alpha)$ is

$$G(x,\alpha) = \left(\frac{1}{A(x)}\right)^\alpha = \left(\frac{1}{1+A(x)-1}\right)^\alpha.$$

$G(x,\alpha)$ can be represented as a composition of two generating functions $R(H(x))$, where

$$R(x) = \left(\frac{1}{1+x}\right)^\alpha = \sum_{n\geq 0}\binom{n+\alpha-1}{n}(-x)^n$$

and

$$H(x) = A(x) - 1.$$

Next we find the coefficients of $H(x)^k$:

$$h(n,k) = [x^n](A(x)-1)^k.$$

Using the binomial theorem, we get

$$h(n,k) = [x^n]\sum_{j=0}^{k}\binom{k}{j}A(x)^j(-1)^{k-j}.$$

Then

$$h(n,k) = \sum_{j=0}^{k}\binom{k}{j}a(n,j)(-1)^{k-j}.$$

Since $h(0) = 0$, we apply the formula for the composition of generating functions [3]

$$g(n,\alpha) = \sum_{k=1}^{n} H^\Delta(n,k)r(k), \qquad g(0) = r(0),$$

where $H^\Delta(n,k)$ are the coefficients of $H(x)^k$.

Then

$$g(n,\alpha) = \sum_{k=0}^{n} h(n,k)\binom{k+\alpha-1}{k}(-1)^k =$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} a(n,j)(-1)^{k-j} \binom{k+\alpha-1}{k} (-1)^k$$

or

$$g(n,\alpha) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} a(n,j)(-1)^j \binom{k+\alpha-1}{k}.$$

After changing the order of summation, we get

$$g(n,\alpha) = \sum_{j=0}^{n} (-1)^j \binom{j+\alpha-1}{j} a(n,j) \sum_{k=0}^{n-j} \binom{j+k+\alpha-1}{k}.$$

Using the well-known identity [6]

$$\sum_{k=0}^{m} \binom{s+k-1}{k} = \binom{s+m}{m}$$

and replacing $m = n-j$ and $s = j+\alpha$, we obtain

$$\sum_{k=0}^{n-j} \binom{j+k+\alpha-1}{k} = \binom{n+\alpha}{n-j}.$$

Then the desired formula for $a(0) = 1$ is

$$g(n,\alpha) = \sum_{j=0}^{n} (-1)^j \binom{j+\alpha-1}{j} \binom{n+\alpha}{n-j} a(n,j).$$

Next we generalize the obtained formula for the case $a(0) \neq 0$:

$$G(x,\alpha) = \left(\frac{1}{a(0) + A(x) - a(0)} \right)^{\alpha} = a(0)^{-\alpha} \left(\frac{1}{1 + \frac{A(x)}{a(0)} - 1} \right)^{\alpha}.$$

The coefficients for the power of a generating function are defined by

$$[x^n] \left(\frac{A(x)}{a(0)} \right)^k = h(n,k) a(0)^{-k}.$$

Therefore, we arrive at the desired formula

$$g(n,\alpha) = \sum_{k=0}^{n} a(0)^{-k-\alpha} (-1)^k \binom{k+\alpha-1}{k} \binom{n+\alpha}{n-k} a(n,k).$$

□

The formula (2) can be represented as follows:

Theorem 2.2. *An explicit formula for the coefficients of a reciprocal generating function $G(x)$ raised to the power of α is equal to*

(3) $$g(n,\alpha) = (n+1) \binom{n+\alpha}{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(k+\alpha) a(0)^{k+\alpha}} \binom{n}{k} a(n,k).$$

Proof. Using

$$\binom{k+\alpha-1}{k}\binom{n+\alpha}{n-k} = \binom{k+\alpha-1}{k}\binom{n+\alpha}{k+\alpha} =$$
$$= \frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)}\frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)\Gamma(n-k+1)} =$$
$$= \frac{\Gamma(k+\alpha)\Gamma(n+\alpha+1)}{(k+\alpha)k!\Gamma(\alpha)\Gamma(k+\alpha)(n-k)!} = \frac{(n+\alpha)\Gamma(n+\alpha)}{(k+\alpha)\Gamma(\alpha)k!(n-k)!} =$$
$$= \frac{n+1}{k+\alpha}\binom{n+\alpha}{n+1}\binom{n}{k}$$

for the formula (2) we arrived at the desired result. □

Corollary 2.3. *For a generating function* $A(x) = \sum_{n\geq 0} a(n)x^n$, $a(0) \neq 0$, *the coefficients of a reciprocal generating function* $G(x) = \sum_{n\geq 0} g(n)x^n$ *are defined by*

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} a(n,k),$$

where

$$a(n,k) = [x^n]A(x)^k.$$

3. Application

3.1. **Generating function** $G(x) = \frac{1}{\sqrt{A(x)}}$**.** For the case $\alpha = \frac{1}{2}$, the formula (3) is

$$g(n) = (n+1)\binom{n+\frac{1}{2}}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{(k+\frac{1}{2})a(0)^{k+\frac{1}{2}}}\binom{n}{k}a(n,k).$$

Since

$$\binom{n+\frac{1}{2}}{n+1} = \frac{1}{2\,4^n}\binom{2n+1}{n+1}$$

we have

$$g(n) = \frac{(n+1)}{4^n}\binom{2n+1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)a(0)^{k+\frac{1}{2}}}\binom{n}{k}a(n,k). \tag{4}$$

Suppose we have the generating function $G(x) = \sqrt{1-x}$. Then for the generating function

$$A(x) = \frac{1}{1-x},$$

the coefficients $a(n,k)$ are

$$a(n,k) = \binom{n+k-1}{n}.$$

From one side the coefficients of $G(x) = \sqrt{1-x}$ are defined by

$$[x^n]\sqrt{1-x} = 1 - \frac{2}{4^n n}\binom{2n-2}{n-1}.$$

From (4) the coefficients are

$$g(n)=\frac{(n+1)}{4^n}\binom{2n+1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\binom{n+k-1}{n}.$$

Then we have the following identity:

$$\frac{(n+1)}{4^n}\binom{2n+1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}\binom{n}{k}\binom{n+k-1}{n}=-\frac{2}{4^n n}\binom{2n-2}{n-1}$$

or

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}\binom{n+k-1}{n}}{2k+1}=\frac{1}{1-4n^2}.$$

For the generating function $G(x)=\sqrt{\frac{x^2}{(\exp(x)-1)^2}}$, we arrive at another identity

$$\sum_{k=0}^{n}\frac{(-1)^k(2k)!\binom{n}{k}S(n+2k,2k)}{(2k+1)(n+2k)!}=\frac{2\,4^nB(n)(n+1)!}{(2n+2)!},$$

where $S(n,k)$ are the Stirling numbers of the second kind and $B(n)$ are the Bernoulli numbers.

3.2. **Generating function** $G(x)=(1+x)^{-\alpha\beta}$**.** Suppose we have the generating function $G(x)=\left(\frac{1}{(1+x)^\beta}\right)^\alpha$. Then for the generating function

$$A(x)=(1+x)^\beta,$$

the coefficients $a(n,k)$ are

$$a(n,k)=\binom{\beta k}{n}.$$

From one side the coefficients of $G(x)=(1+x)^{-\alpha\beta}$ defined by

$$\binom{-\alpha\beta}{n}=(-1)^n\binom{n+\alpha\beta-1}{n}.$$

From (3) we have the following identity

$$(n+1)\binom{n+\alpha}{n+1}\sum_{k=0}^{n}\frac{(-1)^k\binom{\beta k}{n}\binom{n}{k}}{k+\alpha}=(-1)^n\binom{n+\alpha\beta-1}{n}.$$

Then

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{\beta k}{n}\binom{n}{k}}{k+\alpha}=(-1)^n\frac{(\alpha\beta)_n}{\alpha(1+\alpha)_n}.$$

3.3. **Expansion for** $(\Gamma(1+x)\Gamma(1-x))^\alpha$**.** The coefficients for the generating function

$$(\sin(x))^k=\sum_{n\ge k}T_s(n,k)x^n$$

are defined by

$$T_s(n,k)=\frac{1+(-1)^{n-k}}{2^kn!}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(2i-k)^n\binom{k}{i}(-1)^{\frac{n+k}{2}-i}.$$

Then

$$\left(\frac{\sin(\pi x)}{\pi}\right)^k = \sum_{n\geq k} \pi^{n-k} T_s(n,k) x^n.$$

Therefore, applying the formula (3), we get the following expansion:

$$(x\Gamma(1+x)\Gamma(1-x))^\alpha =$$

$$= \sum_{n\geq 0} (2n+1)\pi^{2n} \binom{n+\alpha}{2n+1} \sum_{k=0}^{2n} \frac{(-1)^k}{k+\alpha} \binom{2n}{k} T_s(2n+k,k) x^{2n}.$$

3.4. **Norlund polynomials.** The Norlund polynomials are defined by the exponential generating function [7]

$$\left(\frac{x}{e^x-1}\right)^\alpha = \sum_{n\geq 0} B_n^{(\alpha)} \frac{x^n}{n!}.$$

From

$$\left(\frac{e^x-1}{x}\right)^k = \sum_{n\geq 0} S(n+k,k) \frac{k!}{(n+k)!} x^n,$$

where $S(n,k)$ are the Stirling numbers of the second kind, using the formula (3), we get the following explicit formula for the Norlund polynomials:

$$B_n^\alpha = (n+1)\binom{n+\alpha}{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(k+\alpha)\binom{n+k}{n}} \binom{n}{k} S(n+k,k).$$

3.5. **Bernoulli numbers of the second kind.** The Bernoulli numbers of the second kind are defined by the exponential generating function [7]

$$G(x,\alpha) = \left(\frac{x}{\log(1+x)}\right)^\alpha = \sum_{n\geq 0} b(n,a) \frac{x^n}{n!}.$$

Using the formula (3) and from

$$\left(\frac{\log(1+x)}{x}\right)^k = \sum_{n\geq 0} s(n+k,k) \frac{k!}{(n+k)!} x^n,$$

where $s(n,k)$ are the Stirling numbers of the first kind, we get the following explicit formula for the Bernoulli numbers of the second kind:

$$b(n,\alpha) = \sum_{k=0}^{n} (-1)^k \binom{k+\alpha-1}{k} \binom{n+\alpha}{n-k} \binom{n+k}{k}^{-1} s(n+k,k)$$

or

$$b(n,\alpha) = (n+1)\binom{n+\alpha}{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{(k+\alpha)\binom{n+k}{n}} \binom{n}{k} s(n+k,k).$$

3.6. **Compositional inverse generating function** $xG(x)^{-\alpha}$**.** By the compositional inverse generating function we mean the following:

Definition 3.1. A compositional inverse $\overline{A(x)}$ of generating function $A(x) = \sum_{n>0} a(n)x^n$ with $a(1) \neq 0$ is a formal power series such that satisfies the following condition:

$$A(\overline{A(x)}) = x.$$

Also a compositional inverse generating function can be written as $A^{[-1]}(x)$ or $RevA$.

Suppose we have a generating function $A(x)$ such that $a(1) \neq 0$ and a formula for $a(n,k)$, where

$$A(x)^k = \sum_{n\geq 0} a(n,k)x^n.$$

Then the coefficients of a reciprocal generating function of $G(x)^{\alpha m} = \left(\frac{1}{A(x)}\right)^{\alpha m}$ are defined by

$$g^\alpha(n,m) = (n+1)\binom{n+m\alpha}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{(k+\alpha m)a(0)^{k+\alpha m}}\binom{n}{k}a(n,k).$$

Then using the Lagrange inversion formula [8], we obtain

$$a^{[-1]}(n,m,\alpha) = \frac{k}{n}g^\alpha(n-m,n)$$

or

(5) $$a^{[-1]}(n,m,\alpha) =$$

$$= \frac{m}{n}(n-m+1)\binom{n(1+\alpha)-m}{n-m+1}\sum_{k=0}^{n-m}\frac{(-1)^k}{(k+\alpha n)a(0)^{k+n\alpha}}\binom{n-m}{k}g(n-m,k).$$

Example. Let

$$G(x,\alpha) = \left(\frac{e^x-1}{x}\right)^\alpha = \sum_{n\geq 0} B_n^{(-\alpha)}\frac{x^n}{n!},$$

then we will find an explicit formula for the coefficients of a compositional inverse generating function.

For that we write an expression for the powers of the generating function

$$\left(\frac{e^x-1}{x}\right)^k = \sum_{n\geq 0} T(n,k)x^n,$$

where

$$T(n,k) = \frac{k!}{n!}S(n+k,k).$$

Applying the formula (5), we obtain the following formula

$$g^{[-1]}(n,m) = \frac{m}{n(n-m)!}B_{n-m}^{(n\alpha)} = \frac{m(n-m+1)}{n}\binom{n+n\alpha-m}{n-m+1}\times$$

$$\times\sum_{k=0}^{n-m}\frac{(-1)^k}{k+\alpha n}\binom{n-m}{k}\frac{k!}{(n-m)!}S(n-m+k,k).$$

Using a composition of a generating function $G(x)$ and it's compositional inverse generating function we get the following identities

$$\sum_{k=m}^{n} \frac{kB_{n-k}^{(n\alpha)}}{n(n-k)!}\frac{B_{k-m}^{(-m\alpha)}}{(k-m)!} = \delta(n,m),$$

$$\sum_{k=0}^{n-m} \frac{m+k}{n}\binom{n-m}{k} B_{n-m-k}^{(n\alpha)} B_{k}^{(-m\alpha)} = \delta(n,m),$$

$$\sum_{k=0}^{n-m} \frac{m}{m+k}\binom{n-m}{k} B_{n-m-k}^{(-(k+m)\alpha)} B_{k}^{((m+k)\alpha)} = \delta(n,m).$$

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